PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY, FEBRUARY 22, 2024, ON TAYLOR SERIES, TAYLOR POLYNOMIALS, and CURVES DESCRIBED BY PARAMETRIC EQUATIONS

CLASS #12

NOTATION CONVENTION

Given function $f(x)$,

$f'(x) =$ the first derivative of f

$f^{(1)}(x) = f'(x)$ Also, $f^{(0)}(x) = f(x)$

$f^{(2)}(x) = f''(x)$

$f^{(3)}(x) = f'''(x)$

$\vdots$

$f^{(n)}(x) =$ the n^{th} derivative of f

<u>Definition</u> : For a real number $\underline{a}$ and a differentiable function $f(x)$,

The "<u>Taylor Series for f centered at a</u>" is

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

For those x in the I.O.C., summation $= f(x)$.

When $a = 0$, This is called the "MacLaurin Series for f".

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

The MacLaurin Series for $f(x) = e^x$

$$f(x) = e^x = \frac{f(0)}{0!} + \frac{f'(0)}{1} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Here $f(x) = e^x = f'(x) = f''(x) = f'''(x) = \cdots$.

Since $e^0 = 1$

$$f(x) = e^x = \frac{1}{0!} + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

The MacLaurin Series for $f(x) = e^x$ is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n ,$$ and (from the Ratio Test) $R = \infty$.

Problem: Find the MacLaurin Series for

$$f(x) = x^5 e^{(x^3)} .$$

Sol'n: Since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $R = \infty$, and for $|x| < \infty$, the Series is C,

$$e^{(x^3)} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^3)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n}, \quad |x^3| < \infty, \quad R = \infty$$

$$f(x) = x^5 e^{(x^3)} = x^5 \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^{3n} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{(3n+5)}, \quad R = \infty$$

FINDING the Maclaurin Series for $f(x) = e^x$

Recall, here $a = 0$ and $c_n = \dfrac{f^{(n)}(0)}{n!}$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n	$c_n x^n$
0	e^x	$e^0 = 1$	$e^0/0! = 1$	$1 = x^0/0!$
1	e^x	$e^0 = 1$	$e^0/1! = 1$	$x = \frac{x^1}{1!}$
2	e^x	$e^0 = 1$	$\frac{e^0}{2!} = 1/2!$	$\frac{1}{2}x^2 = x^2/2!$
3	e^x	$e^0 = 1$	$e^0/3! = 1/3!$	$\frac{1}{6}x^3 = x^3/3!$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The Maclaurin series for $f(x) = e^x$ is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n \text{ where } a_n = \frac{x^n}{n!}$$

To find the RADIUS OF CONVERGENCE R, we use the ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1}.$$

For any x in $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1, \text{ so the}$$

Maclaurin Series for $f(x) = e^x$ converge absolutely for all x in $\mathbb{R}$.

The Radius of Convergence is $R = \infty$.

1/ <u>Finding the Maclaurin Series for $f(x) = \cos x$</u>

The series $\sum_{n=0}^{\infty} c_n x^n$ has $c_n = \dfrac{f^{(n)}(0)}{n!}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	c_n	$c_n x^n$	
✓ 0	$\cos x$	1	$1/0! = 1$	$1 = x^0/0!$	$m=0$
1	$-\sin x$	0	$0/1! = 0$	$\cancel{0x}$	
✓ 2	$-\cos x$	-1	$\frac{-1}{2!} = -\frac{1}{2}$	$-x^2/2$	$m=1$
3	$\sin x$	0	$0/3! = 0$	$\cancel{0x^3}$	
✓ 4	$\cos x$	1	$1/4!$	$+x^4/4!$	$m=2$
5	$-\sin x$	0	$0/5! = 0$	$\cancel{0x^5}$	
✓ 6	$-\cos x$	-1	$-1/6!$	$-x^6/6!$	$m=3$
7	$\sin x$	0	$0/7! = 0$	$\cancel{0x^2}$	
✓ 8	$\cos x$	1	$1/8!$	$+x^8/8!$	$m=4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$\cos x = \underbrace{\frac{x^0}{0!}}_{m=0} + \underbrace{(-1)\frac{x^2}{2!}}_{m=1} + \underbrace{\frac{x^4}{4!}}_{m=2} + \underbrace{(-1)\frac{x^6}{6!}}_{m=3} + \underbrace{\frac{x^8}{8!}}_{m=4} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} \xrightarrow{\text{Write using } n} \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

FINDING the TAYLOR SERIES OF $f(x) = \sin x$ Centered at $a = \pi/2$

Here $a = 2$, $c_n = \dfrac{f^{(n)}(\pi/2)}{n!}$ and $c_n (x-a)^n = \dfrac{f^{(n)}(\pi/2)}{n!} (x - \frac{\pi}{2})^n$

Use k:	n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$	$c_n (x-a)^n$
$k=0$	0	$\sin x$	$\sin(\pi/2) = 1$	$\frac{1}{0!}(x-\frac{\pi}{2})^0 = 1$
	1	$\cos x$	$\cos(\pi/2) = 0$	———
$k=1$	2	$-\sin x$	$-\sin(\pi/2) = -1$	$\frac{-1}{2!}(x-\pi/2)^2$
	3	$-\cos x$	$-\cos(\pi/2) = 0$	———
$k=2$	4	$\sin x$	$\sin(\pi/2) = 1$	$\frac{1}{4!}(x-\pi/2)^4$
	5	$\cos x$	$\cos(\pi/2) = 0$	———
$k=3$	6	$-\sin x$	$-\sin(\pi/2) = -1$	$-1/6!\,(x-\pi/2)^6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x - \pi/2)^{2k}$$

It can be shown that the radius of convergence is $R = \infty$.

To Memorize

Table 1
Important Maclaurin Series and Their Radii of Convergence

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$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \dbinom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!} x^2 + \dfrac{k(k-1)(k-2)}{3!} x^3 + \cdots$	$R = 1$

NOT NEEDED

Problem: Find the Sum of the Series

A HINT FOR A HOMEWORK PROBLEM

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\,5^n} = S$$

Sol'n

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\,5^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{5}\right)^n}{n}$$

Since $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$,

$$S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{5}\right)^n}{n} = \ln\left(1+\frac{3}{5}\right) = \ln\left(\frac{8}{5}\right) = S$$

A New and Related Definition :

Defn: For a differentiable function $f(x)$ and a real number a and a positive integer n, "The n^{th} Taylor Polynomial $T_n(x)$ of f (centered at a)" is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

We also have the "Remainder Function $R_n(x)$":

$$R_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k ,$$

So, $f(x) = T_n(x) + R_n(x)$ and

$R_n(x) =$ The ERROR in $T_n(x) \approx f(x)$

Here is an example:

Problem: Let $f(x) = \frac{1}{x^3} = x^{-3}$.

Determine the $n=2$ Taylor Polynomial, $T_2(x)$, for $f(x) = \frac{1}{x^3}$ centered $a = 1$.

Sol$_n$:

$$T_2(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$

Here $a=1$

$$T_2(x) = \frac{f(1)}{0!} + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$f(x) = x^{-3} = \frac{1}{x^3}$

$f'(x) = -3x^{-4} = \frac{-3}{x^4}$

$f''(x) = 12x^{-5} = \frac{12}{x^5}$

$f(1) = 1 = f(a)$

$f'(1) = -3$

$f''(1) = 12$

$$T_2(x) = \frac{1}{0!} + \frac{(-3)}{1!}(x-1) + \frac{12}{2!}(x-1)^2$$

$$\boxed{T_2(x) = 1 - 3(x-1) + 6(x-1)^2}$$

for $f(x) = \frac{1}{x^3}$ with $n=2$ and $a=1$

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Figure 1

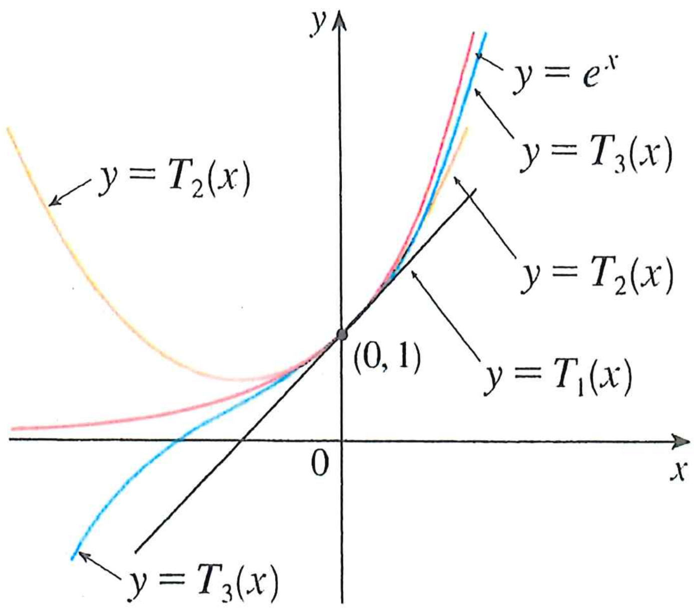

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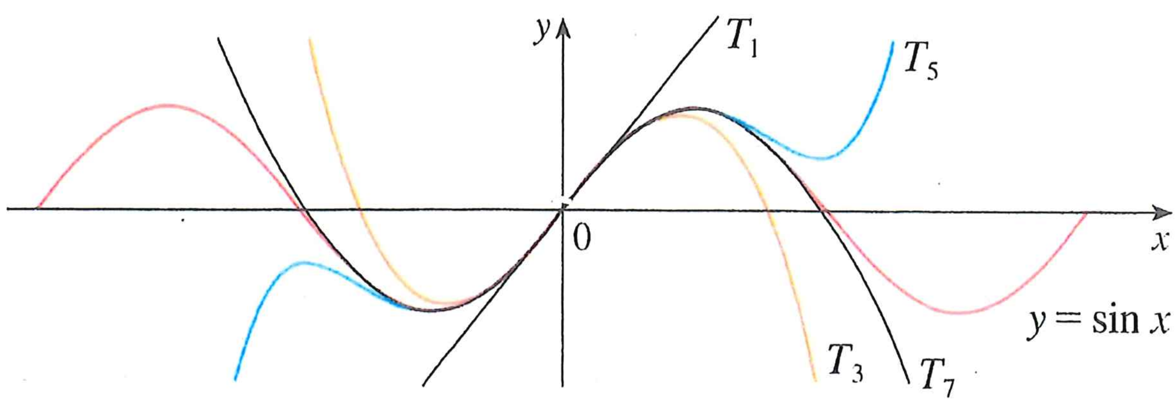
Figure 6



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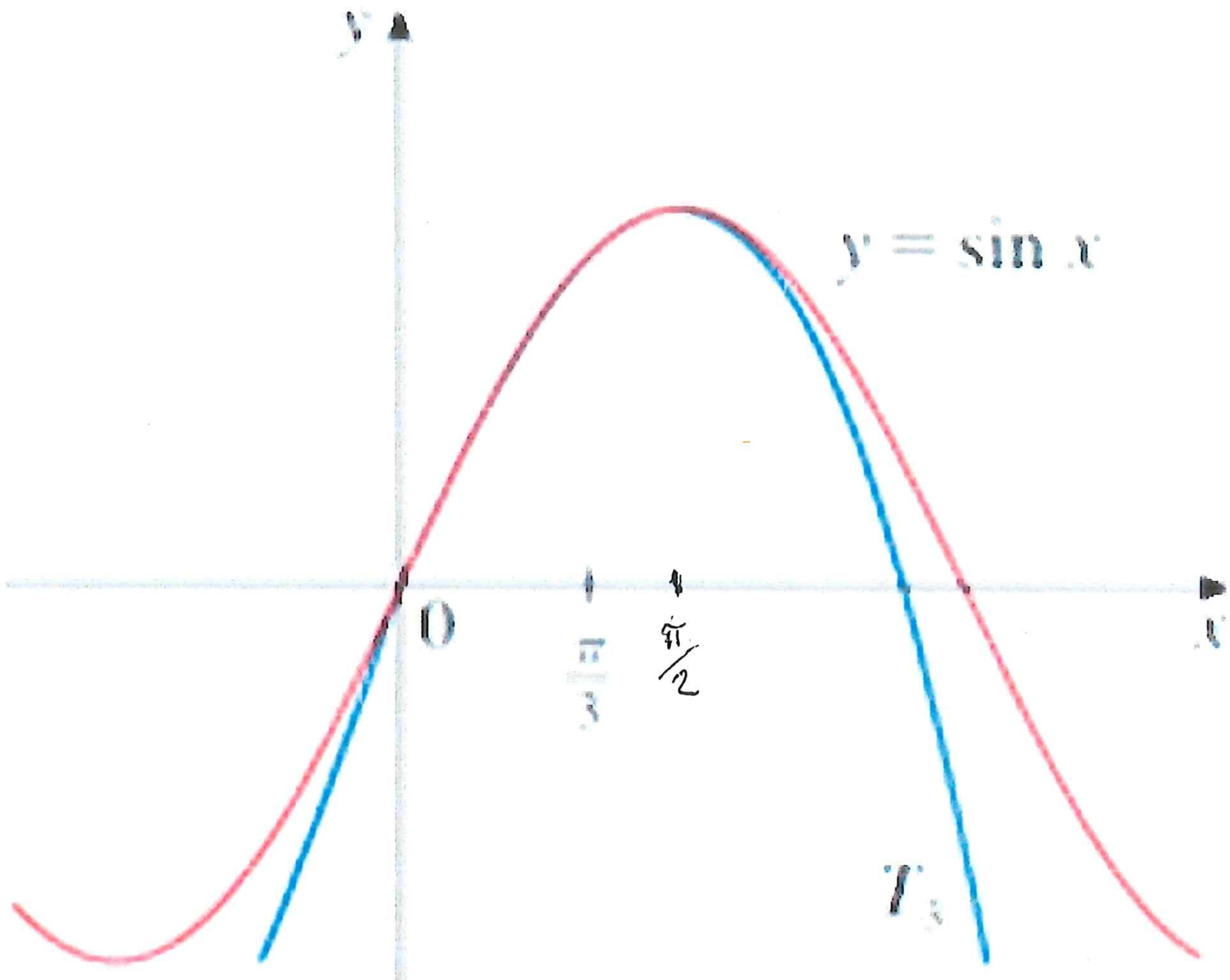

FIGURE 3

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Now, $T_2(x) \approx \frac{1}{x^3}$ for x-values close to $a = 1$.

What is the error in $T_2(1.1) \approx f(1.1) = \frac{1}{(1.1)^3}$?

Soln: $f(x) = \frac{1}{x^3}$ here.

$$T_2(x) = 1 - 3(x-1) + 6(x-1)^2$$

$$T_2(1.1) = 1 - 3(0.1) + 6(0.1)^2$$

$$T_2(1.1) = 1 - 0.3 + 0.06 = \underline{0.76}$$

$$T_2(1.1) = 0.76$$

$$\frac{1}{(1.1)^3} = \frac{1}{1.331} = 0.751314801$$

The Error in $T_2(1.1) \approx \frac{1}{(1.1)^3} = f(1.1)$

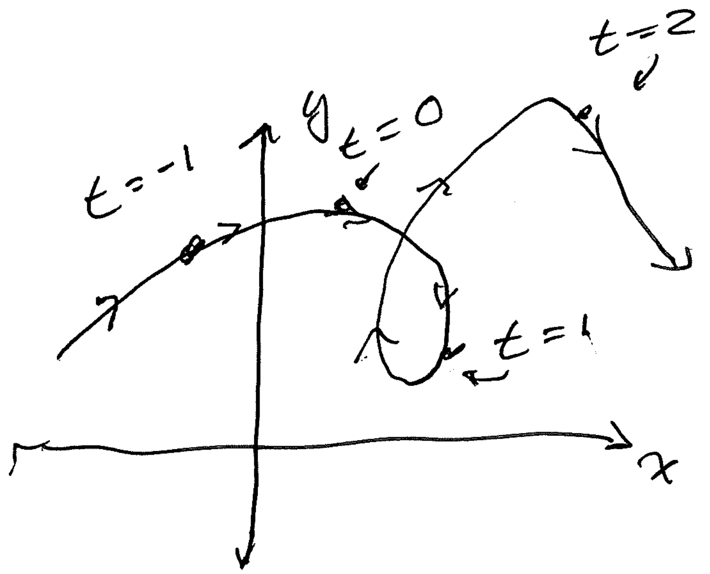
is $|0.76 - 0.751314801|$

ERROR $= 0.008685199$.

Describing Curve with Parametric Equations

A particle travels along a path in the xy-plane such that, at time t seconds, its position in the plane is $P(t) = (x, y)$

$P(t) = (x, y) = (f(t), g(t))$



The Equations

$x = f(t)$
$y = g(t)$

Parametric EQUATIONS and t is the parameter.

Sometime we write

$x = x(t)$
$y = y(t)$

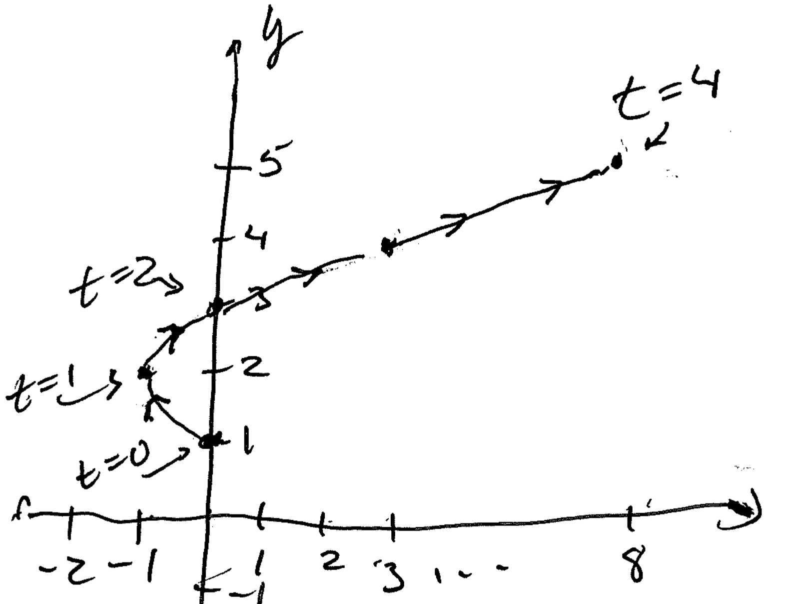
Ex: Sketch the curve given by the Equations

$x = f(t) = t^2 - 2t$

$y = g(t) = t + 1$ where $0 \leq t \leq 4$

Point$(x, y) = P(t)$

t	$(t^2-2t, t+1)$
0	$(0, 1)$
1	$(-1, 2)$
2	$(0, 3)$
3	$(3, 4)$
4	$(8, 5)$



The Arrow heads indicate "The direction of increasing values of the parameter."

Ex: $x = t^2 - 2$ No restrictions on t.

$y = t + 1$